The liar paradox, which is also known as “This sentence is a lie” or “I am lying”, switches between true and false continuously. If the sentence is true, then the sentence states that it is false. If false, then the statement “This sentence is false.” makes it true again. And so on, indefinitely. A paradox is defined as a seemingly absurd or contradictory statement or proposition which when investigated may prove to be well founded or true. (Source: https://www.lexico.com/en/definition/paradox) Mathematicians shy away from paradoxes because of their often inherent contradictions. But Spencer-Brown found a way to deal with seemingly contradictory statements. The paradox “This sentence is false.” can be rephrased as a recurrent form expression.

$$TSIF = \overline{TSIF}|$$

Note that the expression TSIF, short for “This sentence is false.”, occurs both on the Left Hand Side (LHS) and the Right Hand Side (RHS) of the equal (=) sign: LHS = RHS. This means that LHS (= RHS) can be substituted in the occurrence of LHS in the RHS leading to a recursion

$$TSIF = \overline{TSIF}|$$

$$= \overline{\overline{TSIF}|}|$$

$$= \overline{\overline{\overline{TSIF}|}|}|$$

$$= \overline{\overline{\overline{\ldots}|}|}|$$

which is equivalent to an alternating time sequence

$$TSIF = \cdots, \rceil, \overline{\rceil}|, \rceil, \overline{\rceil}|\rceil, \overline{\rceil}|, \ldots = \cdots, marked, unmarked, marked, unmarkerd, \ldots$$

In a sense, the recurrent form can be regarded as a form expression that is re-entered in its own indicational space. And as a result, an oscillation my occur switching between the marked (true) and unmarked (false) state. Spencer-Brown devised a special symbol for re-entrance.

$$f = \overline{f}| \quad \stackrel{\text{def}}{=} \quad \underline{\,\lrcorner\overline{\;}|\,} \quad \text{with f denoting an arbitrary expression, e.g., TSIF}$$

So, Spencer-Brown discovered that the solution of paradoxical statement is an oscillation. That is to say, besides the markedness and unmarkedness sides of a distinction in space, a new dimension is introduced: time! Paradoxes are de-paradoxified in time. At one moment in time a statement may be true, while at another moment a statement may be false, which makes perfectly sense.

The impact of this discovery cannot be overstated. The idea of a paradox leading to an oscillation in time is key to understanding self-production (autopoiesis) in living organisms to sustain life as shown by Maturana and Varela (cite{?}). Luhmann applied this idea to social systems in his autopoietic turn to show how societies carry on (cite{?}).

Now consider this equation.

$$x^2 = -1$$

It is well-known that this equation has no real solution because squaring a negative or a positive real number always yields a positive real number, so it can never be -1. The equation can be rewritten as a recursive expression, in which *x* is occurring in both the LHS and RHS of the equal (=) sign.

$$x = \frac{-1}{x}$$

This equation has the same paradoxical qualities of the liar paradox. Obviously, If a solution would exist, this can only be with *x* taken on the unity value of +1 or -1. But unfortunately, when +1 or -1 is substituted for x, the result is precisely the opposite, like an oscillation.

$$+1 = \frac{-1}{+1} = -1$$

$$-1 = \frac{-1}{-1} = +1$$

Perhaps unknown for readers not well-acquainted with mathematics, the equation

$$x^2 = -1$$

can be solved by resorting to imaginary numbers. By definition

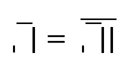
$$i^2 = -1$$

with

$$i = \sqrt{-1}$$

How strange it may look, imaginary numbers are as real as real numbers. It took a while to get accustomed with imaginary numbers, but that was once also the case with zero and negative numbers. Nowadays, they are applied routinely in all kind of engineering domains.

Spencer-Brown notes that the imaginary number *i* can be regarded as an expression that oscillates endlessly between the values +1 and -1, and by analogy introduces the idea of an imaginary logical value that oscillates between marked and unmarked (cite{Art Collins, blz. 89}). Varela elaborated on this idea and extended Spencer-Brown two-valued system (marked and unmarked) to include an imaginary, oscillatory state. The mark of distinction severs a space in two sides of which one side is marked and the other one is unmarked. Clearly, the imaginary state cannot be associated with one of these. That leaves the mark of distinction itself as the only place for positioning the imaginary state. For this reason, the imaginary state is called the boundary state, which is neither marked nor unmarked. Varela devised a three-valued mathematical system in which the following equation holds (cite{?}).



This should be interpreted as when crossing the mark from the boundary state one enters a new state that is also the boundary state. Varela made an important contribution by making the boundary state a first-class citizen in his calculus, which was later in collaboration with Kauffman extended to a four-valued system to study waveforms (cite{waveforms}). Collins devised a four-valued system as the logical counterpart of complex numbers containing a real and an imaginary part (cite{Collins}).

The boundary state captures nicely the idea of self-producing by means of self-reference. Starting from the boundary, a new boundary is established. In the realm of a living organism, an organism renews itself by using its own elements to produce new elements surrounded by a new boundary.

Principle: Embrace the paradox, i.e., a difference in what was previously stated and therefore contradicting what was said before. Differences keep setting things in motion. Without differences we cease to exist. Therefore, change is inevitable, in fact, it is a necessity for living.

Closely related to the notions of self-reference and self-producing (autopoiesis) are the concepts of autonomy and closure. With these four notions, a new system concept can be conceived. Instead of defining a system as a set of interconnected elements that performs an input-output transformation, a system can be seen as a self-producing entity. By the way, this is the system definition given by Luhmann, again in the form of a paradoxical, self-referential expression. A system is defined as the difference between the system and its environment. This can be written in LoF terms as follows.

$$system \stackrel{\text{def}}{=} \overline{system|}\,environment$$

Or alternatively, the same definition can be expressed with the boundary state mark.

$$\underline{\overline{\overline{system|}\,environment|}} = \overline{\overline{\overline{\overline{\ldots system|}\,environment|}\,system|}\,environment|}$$

The system is contained within the confinement of the first distinction. Therefore the system can be seen as an autonomously operating closed entity producing new elements using its own elements. This definition of a system is self-referential. The distinction between the system and its environment is re-entered in its own indicational space and as a result a new system is created with a newly established boundary with its environment. The re-entrance of the distinction between the system and its environment can be seen as feedback providing information to determine the next step to be taken by the system. This is done autonomously of course, the system's destiny is controlled by the system itself. A system conceived in this way is said to be operationally closed (i.e., operating autonomously) and structurally open (i.e., vitalized by its environment).

Interestingly, the distinction between observer - who makes a distinction - and what is observed – the indication of a distinction - becomes obscured. They are in fact the same. The observer, which is the (human) system itself, makes a distinction. But according to Spencer-Brown, there cannot be a distinction without an indication and vice versa, they arise together. The indication itself is a distinction in its own right indicating the difference between system and environment. This means that the act of distinction is necessarily circular: you, as an observer, make a distinction, which is you, to indicate a difference of what is being observed via the mark of distinction, which is you again. This leads to the following recursive definition:

$$Indication = Distinction \xrightarrow{implies} Indication$$

Because distinction and indication co-arise, this definition can be written equally well as:

$$Distinction = Indication \xrightarrow{implies} Distinction$$

The coincidence of the observer and the observed plays an import role in second-order cybernetics and Luhmann's social theory. It leads to the notion of second-order observations to focus on *how* an observer observes instead of *what* an observer observes. In fact this notion was that important to Luhmann that after his autopoietic turn he took the observation turn.

Hier hoort nog een plaatje bij.

The last sentence in Spencer-Browns's LoF captures this all as follows.

> We see now that the first distinction, the mark, and the observer are not only interchangeable, but, in the form, identical.