Cybernetics & Human Knowing, Vol.8, no.1–2, 2001, pp. 133–140

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American Society for Cybernetics
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science of human understanding

On the Cybernetics of Fixed Points

Louis H. Kauffman

In his paper "Objects as Tokens for Eigenbehaviours" [2] Heinz von Foerster suggests that we think seriously about the mathematical structure behind the constructivist doctrine that *perceived worlds are worlds created by the observer.* At first glance such a statement appears to be nothing more than solipsism. At second glance, the statement appears to be a tautology, for who else can create the rich subjectivity of the immediate impression of the senses? At third glance, something more is needed. A beginning in that direction occurs with Heinz' paper. In that paper he suggests that the familiar objects of our experience are the fixed points of operators. These operators *are* the structure of our perception. To the extent that the operators are shared, there is no solipsism in this point of view. It is the beginning of a mathematics of second order cybernetics.

Where are these operators and where are their fixed points? Lets start back closer to the beginning. Wittgenstein says, at the beginning of the Tractatus [3],

"The world is everything that is the case." What is the case are the distinctions, including the distinction that there is a world at all. It is tempting to succumb to the idea that behind this tapestry of distinction there is a hidden inner mechanism of the "thing in itself" hiding behind a world of appearances. That "thing in itself" is the other side of the distinction that is world of appearances. One can take the point of view that the world is the world of appearances. But one can take the agnostic point of view that a distinction can be deeply investigated from one of its sides without a belief in the existence of an unobservable side. It is, I believe, this agnostic point of view that leads directly to objects as tokens for eigenbehaviours.

For consider the relationship between an observer O and an "object" A. The key point about the observer and the object is that "the object remains in constant form with respect to the observer". This constancy of form does not preclude motion or change of shape. Form is more malleable than the geometry of Euclid. In fact, ultimately the form of an "object" is the form of the distinction that "it" makes in the space of our perception. In any attempt to speak absolutely about the nature of form we take the form of distinction for the form. (parphrasing Spencer-Brown [1]). It is the form of distinction that remains constant and produces an apparent

object for the observer. How can you write an equation for this? The simplest route is to write

O(A) = A.

The object **A** is a fixed point for the observer **O**. The object is an eigenform. We must emphasize that this is the most schematic possible description of the condition of the observer in relation to an object **A.** We only record that the observer as an actor (operator) manages through his acting to leave the (form of) the object unchanged. This can be a recognition of the symmetry of the object but it also can be a description of how the observer, searching for an object, makes that object up (like a good fairy tale) from the very ingredients that are the observer herself. This is the situation that Heinz von Foerster has been most interested in studying. As he puts it, if you give a person an undecidable problem, then the answer that he gives you is a description of himself. And so, by working on hard and undecidable problems we go deeply into the discovery of who we really are. All this is symbolized in the little equation **O(A) = A.**

And what about this matter of the object as a token for eigenbehaviour? This is the crucial step. We forget about the object and focus on the observer. We atttempt to "solve" the equation **O(A) = A** with **A** as the unknown. Not only do we admit that the "inner" structure of the object is unknown, we adhere to whatever knowledge we have of the observer and attempt to find what such an observer could observe based upon that structure.

The rest of the paper is a multi-logue about the attempts to solve the equation of the obsever in relation to his/her observation. We first encounter Mr. D, who has solved his own equation in such a way that he has no head and instead has a great open space of possibility where his head was supposed to be. This requires a drink to ingest and we go to Zermelo's Bar, where we find two mathematicians arguing over the solution to an equation whose solution is the Golden Ratio, a proportion well known to the Greeks. The mathematicians are a little hard to follow, but their discussion turns on all the essential issues of recursion, reality and infinity that we will need for this adventure. Then Dr. Von F appears in the bar (we think you can guess who this is) and explains the nature of eigenforms. He is followed by a character named Charlie and Dr. CC, a linguist and logician, then by Dr. HM, a biologist. Later there appears a physicist, Dr. JB and finally Dr. R himself, the source of the self-referential paradox. We hope that you will join in on this discussion yourself.

Infinite Recursion and Its Relatives

Our problem is to solve the equation O(A) = A for A in terms of O.

For example, suppose that the observer O is Mr. D, a man who insists that he has no head. We interview him. Well Mr. D, why do you say that you have no head? Mr. D. replies. Oh it is so simple, you will see at once what I mean. In fact, consider what you yourself see. Look directly around. Do you see your head? No. You see and feel a great open space of perception where your head is supposed to be, and a flow of thoughts and feelings. But no head! The body comes in.

Shoulders, arms, legs, shoes and the world. But no head. Instead of a head there is a great teeming void of perception. Once I realized this, I knew that the relationship of a self to reality was indeed deep and mysterious.

As we can see, Mr. D has discovered that what is constant for his visual observer is a body without a head. He has solved the problem of finding himself as a solution of the equation of himself in terms of himself. Perhaps we need a drink.

We walk into Zermelo's Bar and two mathematicians appear on the scene. One says to other: How do you solve this equation? I want a postive real solution.

$$\mathbf{1 + 1/A = A.}$$

The second one says: Nothing to it, we multiply both sides by the unknown **A** and rewrite as

$$\mathbf{A + 1 = A^2.}$$

Then, solving the quadratic equation, we find that

$$\mathbf{A = (1 + \sqrt{5})/2.}$$

The first mathematician says: Nice tricks you have there, but I prefer infinite reentry of the equation into itself. Look here: If **A = 1 + 1/A**, then

$$\mathbf{A =}$$
$$\mathbf{1 + 1/A =}$$
$$\mathbf{1+1/(1+1/A) =}$$
$$\mathbf{1+1/(1+1/(1+1/A)) =}$$
$$\mathbf{1+1/(1+1/(1+1/(1+1/A)))}$$

and I will take this reentry process to infinity and obtain the form

$$\mathbf{A = 1+ 1/(1+1/(1+1/(1+1/(1+ 1/(1+1/(1+...))))))).}$$

The second mathematician then says: Well I like your method. We can combine our answers and write a beautiful formula!

$$\mathbf{(1+\sqrt{5})/2 =}$$
$$\mathbf{1+ 1/(1+1/(1+1/(1+1/(1+ 1/(1+1/(1+...))))))}$$

Why do you like this formula? says the second guy. Well, sez the first guy, the left hand side is a definite irrational number and it is easy to see by squaring it that it satisfies the equation $\mathbf{A^2 = A + 1}$ as we wanted it. But irrational numbers have a curiously tenuous existence unless you know a way to calculate approximations for them. On the other hand, your right hand side can be regarded as the limit of the fractions

1 = 1/1
1+1/1 = 2/1 = 2
1+1/(1+1/1) = 3/2
1+1/(1+1/(1+1/1)) = 5/3
1+1/(1+1/(1+1/(1+1/1))) = 8/5
1+1/(1+1/(1+1/(1+1/(1+1/1)))) = 13/8
1+1/(1+1/(1+1/(1+1/(1+1/(1+1/1))))) = 21/13

with the first few terms of this limit being

$$\mathbf{(1+\sqrt{5})/2 = 1.618...}$$

On top of this your infinite formula actually does reenter itself as an infinite expression it really is of the form

$$\mathbf{A = 1 + 1/A.}$$

The first guy comes back with: Well it sounds to me like you really believe in the "actual" infinity of the terms on the right-hand side. I also like to imagine that they are all there existing together in space with no time.

Right! says the second guy. We know that this is an idealization, but it lets us reason to correct answers and to put them in an aesthetically pleasing form.

The bartender is listening to all of this, and he leans over and says: You guys have to meet a couple of others on this score. There is Dr. Von F and Dr. CC. They both have some ideas very similar to yours. Hey, here is Dr. Von F now. Dr. Von F, could you tell these fellows about your eigenforms?

Jah! Of course! It is all very simple. We just combine this notion of recursion with the most general possible situation. Suppose we have any observer **O** and we wish to find a fixed point for her. Well then we just let the observer act without limit as in

$$\mathbf{A = O(O(O(O(O(O(O(...)))))))).}$$

After infinity, one more application of **O** does not change the result and we have

$$\mathbf{O(A) = A.}$$

This is very simple, no? And it shows how we make objects. These objects are the tokens of our repeated behaviours in shaping a form from nothing but our own operations. As I have said before, the human identity is precisely the fixed point of such a recursion. "I am the link between myself and observing myself." [2]

The first mathematician makes a comment: What you are doing is a precise generalization of my infinite continued fraction! If I had defined

$$\mathbf{O(A) = 1 + 1/A}$$

then we would have

$$\mathbf{O(O(O(...))) = 1+ 1/(1+1/(1+1/(1+...))).}$$

But I am puzzled by your approach, for it would seem that you are willing that your solution **A** will have no relation with how the process starts, and also it may not be related to the original domain in which it was constructed!

For example, in my mathematics, I could consider the operator

$$\mathbf{O(A) = -1/A}$$

and this operator does not have a fixed point in the real numbers, but if we take **A=i** where $\mathbf{i^2 = -1}$ (the simplest imaginary number), then **O(i)=i.** Are you suggesting that

$$\mathbf{i = -1/-1/-1/... ?}$$

Dr. Von F replies: Jah, Jah! This is very important! The fixed point can be a construction that breaks ground into an entirely new domain! Actually, I am mainly interested in those fixed points that do break new ground. We are looking for the places where new structures emerge. In your mathematics you have illustrated this in two ways. In the first recursion, the values converge to an irrational number (the golden ratio). All the finite approximations are rational

fractions (ratios of Fibonacci numbers) but in the limit of the infinite eigenform, you arrive at this beautiful new irrational number! And in your second example all the finite approximations oscillate like a buzzer, or a paradox, between positive unity and negative unity, but the eigenform is a true representative of the imaginary square root on minus one! And don't forget that this "imaginary" quantity is fundamental to both logic and physics. The fully general eigenforms are fundamental to the ontology of the world.

Suddenly the door to Zermelo's Bar opens and in walks a character that everyone calls "Charlie." Charlie! says the barkeep, where have you been? We have a good discussion on signs going here. You have to hear this stuff. Charlie says, Well I heard just about everything Dr. Von F said as I admit here to a bit of eavesdropping on the other side of the door! These eigenforms of Von F are quite familiar to me as I have thought continuously along these lines for many years. You see, any sign once you look at it in the context of its reference and the continous expansion of its interpretant becomes a growing complex of signs referring to other signs, growing until the references close on themselves and, as Dr. Von F correctly describes, these closures are the eigenforms, the tokens for apparently stable behaviours. As the complex of signs grows, the complex itself is a sign and as the closures occur that sign becomes a sign for itself. We humans are in our very nature such signs for ourselves.

Dr. Von F says: Well I always say that I am the link between myself and observing myself. I am a sign for myself!

At this point Dr. CC chimes in: But Dr. Von F and Charlie, this excursion to recursion and infinity seems quite excessive! It is all right for mathematicians to imagine such a thing, but we humans exist in language and the finiteness of expressions. Surely you do not suggest that this profligate composition of the operator and expansion of sign complexes actually happens!

Well, Dr. CC, says Von F, I am really a physicist and well aware of the speed of physical process in relation to the very slow pace of our verbal thought. Surely you have stood between two facing mirrors and seen the near-instantaneous tunnel of reflections created by light bouncing back and forth between the mirrors. Yes, I am seriously suggesting that the self-composition of the observer is carried to high orders. These orders are sufficiently large and accomplished with such a high speed that they appear infinite in the eyes of the observer. Now you may detect the beginning of a paradoxical flight here. The very observer who is too slow to detect the difference between a large number and infinity is yet so quick and subtle that he/she can produce this flight to infinity. But I beg your pardon, this is still a matter of the interaction of slow thought and fast action. Wave your arm back and forth rapidly in front of your eyes. For all practical purposes the arm appears to be in two places at the same time! You do not deny that it is "you" that moves the arm, and it is "you" that perceives it.

I simply go further and suggest that every perception is based on such an illusion of permanancy, based on the self composition of your self. You do it all and you are surprised at the result — you can not perceive all that you do!

Charlie adds: I agree but do not have to rest on physics. Our shortsighted view of our own nature arises from the difficulty in reckoning that our true nature is as signs for ourselves. It is only at the limit of eigenbehaviours that such signs appear simple. We partake of the complexity of the universe.

Dr CC replies: Ah Charlie and Dr. Von F, I have been working in the linguistic and logical realm and you will see that our points of view are mutually supporting. For I imagine the structure of the observer as a big network of communicating entities. These entities have so much interrelation among themselves that their identities begin to merge into one identity and that is the apparent identity of the self.

Charlie interrupts with: Yes! That is the essence of continuity.

Dr. CC continues. I agree! The infinity in my view is not with any one of them, but with the aggregate of them that has become so large as to begin to merge into a continuity.

But let me explain: If A and B are entities in my "community of the self", then they can interact with each other and with themselves. These processes of interaction produce new entites who exist at the same level as the original entities. Can you imagine this? Of course you can, you are such an entity. For example, I suggest to you that you are the self that thinks kindly of others, that you satisfy the equation **SX = KX** where **S** is "you" and **KX** is the being "thinks kindly of **X**". *Then that entity **S** exists.* In the world of language, *every definable entity exists*. The consequence is that **S** might even think kindly of herself as in **SS = KS**. That **S** can think kindly of herself is, in this linguistic world, dependent on the condition that the kindly thinking observer is an observer at the same level as any other observer. Now there are many such entities. Watch this magic trick. Let

$$\mathbf{GX = O(XX).}$$

The entitity **G** is the observer who observes an entity observing herself. What happens when G observes herself? Then **G** observes herself observing herself and we have a fixed point, an eigenform!

$$\mathbf{GG = O(GG).}$$

I have constructed the eigenform without the infinite composition of the observer upon herself. Of course once this self-reflexive construction comes into the being of language then it runs automatically to the level of practical infinity and produces your recursion.

$$\mathbf{GG = O(GG) = O(O(GG)) = O(O(O(GG))) = ...}$$

I believe my linguistic construction provides the context for your observer's self interaction. The true infinity in my world is a distributed infinity of beings each coming into being as a name for a process of observation. This contines without end and is the basis of the coincidence of the language and the metalanguge in this world.

At this point Dr. HM, a biologist, walks into the room. He remarks: I see that you have been discussing the stability of perceptions from physical and linguistic principles. Let me tell you how I see these matters in my domain. The beings you talk about are biological, not just logical. They exist in the evolutionary flow of

coordinations of coordinations that give rise to the mutual patternings that you call "language" and "thought". It is not at all surprising that each such being, coordinated with the others in the deep flow of its history in biological time will appear layered like an onion with the actions of each on each. The long time history of mutual interaction and coordination will generate the appearance of the eigenforms. But there is no "disembodied observer" who generates these forms from some abstract place. In biology there is no problem of mind (abstract observer) and body. They are one. Mind and observer both refer to the conversational domain that arises in the construction of the coordination of coordinations that is language. The disembodied observer is a fantasy that is convenient for the mathematician or the physicist. In the biological realm all forms are generated through time in an organic way.

And finally, Dr. JB enters the room, a very theoretical physicist. He says: Ah it is not surprising, but you all have the business of objects and eigenforms quite wrong. Let me start with the views of the biologist Dr. HM. You see, there is no time. None. Time is an illusion. Of course in order to tell you about this insight I shall have to use words that appear to describe states in time. That is my fate to be so projected into language. You must forgive me.

Each moment of being is eternal, beyond time. I prefer to call such moments "time capusules." Each moment contains that possibility that it can be interpreted in terms of a "history", a story of events leading up to the "present moment" that constitutes the time capsule as a whole. But this history is a pattern in eternity. That the history can be told with some coherence and that we manage to tell the story of "past events" leads us to believe that these past events "actually happened". But in fact what has happened is happening now and only now in the eternity of the time capsule whose richness dervives from the superposition of its quantum states.

At this point the bartender chimes in: I'll drink to that. Time is a grand illusion and a wee scotch from my bar will convince ye o' that in less time than it takes to wink an eye!

All well and good, says Dr. R, who just walked into the bar, but as I was telling my friend Frege, if there is one thing that will give us trouble it is this notion of eternity and the non-existence of time. For as I told Gottlob just the other day, you have only to imagine the timeless reality of the set of all sets that are not members of themselves and you will have to leave logic behind! I gave up long ago my travails on this issue with Professor Whitehead. We tried to make logic go first and it was a disaster. Now I let logic run along behind and there is no problem at all. As far as fixed points are concerned my favorite is Omega, the set whose only member is Omega herself. You see that the act of set formation is nothing but an act of reflection. Omega finds herself in reflecting on herself.

Dr. CC retorts: Well, Russell, I hardly expected you to capitulate your position on logic. Your Type is hardly likely to just slip away. I prefer to make a specimen of your famous set in the following way. I let **AB** mean that "B is a member of A".

Then I define your set of all sets that are not members of themselves" by the equation

$$\mathbf{Rx = \sim xx.}$$

Then we can pin the specimen to the board by substituting **R** for **x** as in

$$\mathbf{RR = \sim RR.}$$

This **RR** is a fixed point for negation. It is neither true nor false. I do not leave logic behind. I imagine new states of logical discourse that are beyond the true and the false. Your set performs this transition to imaginary Boolean values.

Now Dr. HM says: Well I see you fellows are beginning to foment an argument. I feel that I must point out to you that logical paradox occurs only in the domain of language. There is no such matter as the paradox of the Russell set in the natural domain. In the natural domain, all apparent contradictions are only antimonies in the eyes of some observer. Nature herself runs in the single valued logic of the evolutionary flow. This is why I emphasize that it is only in the linguistic domain of coordinations of coordinations that the eigenforms arise. At the biological level there are processes that can be seen as recursions, but this seeing is already at the level of the coordinations. There is no mystery in this, but it is neccessary to round out the mathematical models with the prolific play and dynamics of the underlying biology.

In this sense biology is prior to physics as well as cognition.

At this point a tremour shakes the bar and the lights go out. I am sorry folks, the bartender says from the darkness, but this is another one of our natural events in the single valued logical flow of biological time — a small earthquake. I will have to ask you to leave now for your own safety. And so the discussion ended, unfinished but perhaps that was for the best.

Notes and a Mathematical Appendix

The story in section 2 presents a number of different points of view about the cybernetics of fixed points. Fixed points can be produced by infinite recursion, by direct self-reference, through the linguistics of lambda calculus, and by approximation to infinites. Mr. D is a fictionalized version of Douglas Harding the man who indeed realized that he did not have a head, and had the courage to write about it. The good Drs. at the bar represent these points of view and are thinly disguised representatives of the viewpoints of Heinz von Foerster, Alonzo Church and Haskell Curry (Dr. CC), Humberto Maturana and the physicists Julian Barbour. Charlie represents the American mathematical philosopher Charles Sanders Peirce. Dr. R. represents Bertrand Russell, the inventor of the set of all sets that are not members of themselves. All this is only the beginning. The most famous fixed point of them all is the Universe herself, acted here by the bartender.

References

1. G. Spencer-Brown, “Laws of Form”, George Allen and Unwin Ltd. London (1969).
2. H. von Foerster, Objects as tokens for eigenbehaviours, in “Observing Systems”, Systems Inquiry Series (1981), pp. 274-285.
3. L. Wittgenstein, “Tractatus Logicus Philosophicus”, Routledge and Kegan Paul, London & NY (1922).